1 Chapter 1

1.1 Rings, Ideals, Radicals

1. Exercise 1. Show that if x is nilpotent and u is a unit, $x + u$ is a unit.
2. Exercise 2. Let $f = a_0 + a_1 x + \cdots + a_n x^n \in A[x]$. Prove that
   (a) f is a unit of $A[x]$ if and only if all the coefficients but the constant term are nilpotents of A and the constant term is a unit of A.
   (b) f is nilpotent if and only if all the coefficients are nilpotent.
   (c) f is a zero-divisor if and only if f is annihilated by a nonzero element of A.
   (d) f is *primitive* if its coefficients generate the unit ideal. Prove that a product is primitive if and only if its coefficients are primitive.

   (Note: if the ring A is a unique factorization domain, the word "primitive" has a slightly different meaning: in that context it means the coefficients do not have a nonunit common factor. The two meanings coincide if the ring is a principal ideal domain.)
3. Exercise 4. Show that in $A[x]$, the Jacobson radical and nilradical are equal.
4. Exercise 6. A ring A has the property that every ideal not in the nilradical contains a nonzero idempotent (i.e. an element x such that $x^2 = x$). Prove that the nilradical and Jacobson radical of A coincide.
5. Exercise 7. Let A be a ring in which all $x \in A$ satisfy $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal of A is maximal.
6. Exercise 8. Let A be a nonzero ring. Show that the set of all prime ideals has elements that are minimal with respect to inclusion.
7. Exercise 10. Let A be a ring, $\mathfrak{N}$ its nilradical. Show the following are equivalent: (i) A has just one prime ideal; (ii) every element of A is either a unit or nilpotent; (iii) $A/\mathfrak{N}$ is a field.
8. Exercise 11. A ring A is *boolean* if $\forall x \in A$, $x^2 = x$. In a boolean ring, show that
   (a) $2x = 0$.
   (b) Exercise 12. Prove that a local ring contains no idempotent $\neq 0, 1$.

1.2 Prime Spectrum

This and the next section set up fundamental tools of algebraic geometry. We gain insight into the geometric objects under study (curves, surfaces, etc.) by looking at the ring of polynomial functions on those objects. We also reverse the process and start with a ring and construct an underlying geometric object of which it can be seen as the "ring of functions." This underlying geometric object is called its *prime spectrum*. The following exercises define the prime spectrum. See the comments below on exercise 16c, and also exercises 26-28, for more context. Also, Exercises 23-24 in Chapter 3 are aimed at fleshing out the way in which it makes sense to think about the ring elements as "functions" on the prime spectrum.

1. Exercise 15. Let A be a ring and let $X = \operatorname{Spec} A$ be the set of prime ideals of A. For arbitrary $E \subset A$, define $V(E)$ to be the set of all prime ideals containing E. Check that

   (a) If $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$.

   (b) $V(0) = X$ and $V(1) = \varnothing$.

   (c) If $(E_i)_{i\in I}$ is a family of subsets of A, then

   $$V\left(\bigcup_{i\in I} E_i\right) = \bigcap_{i\in I} V(E_i)$$

   (d) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

   These results show that sets of the form $V(E)$ are closed under arbitrary intersection and finite union and contain $X, \varnothing$; thus they obey the axioms for the closed sets of a topology; it is called the Zariski topology on $X = \operatorname{Spec} A$.

2. Exercise 16. Describe $\operatorname{Spec} A$ for $A =$

   (a) $\mathbb{Z}$.

   (b) $\mathbb{R}$.

   (c) $\mathbb{C}[x]$.

   (d) $\mathbb{R}[x]$.

   (e) $\mathbb{Z}[x]$.

3. Exercise 17. If $f \in A$, let X_f be the complement of $V(f)$ in $X = \operatorname{Spec} A$. (In the geometric picture based on $A = k[x_1, \dots, x_n]$, X_f is the complement of a hypersurface...) Prove the following:

(a) The X_f form a basis for the Zariski topology.

(b) $X_f \cap X_g = X_{fg}$.

(c) $X_f = \varnothing \Leftrightarrow f$ is nilpotent.

(d) $X_f = X \Leftrightarrow f$ is a unit.

(e) $X_f = X_g$ if and only if (f) and (g) have the same radical.

(f) X is quasicompact. (Aside: in algebraic geometry, the word "compact", meaning, as usual, that every open cover has a finite subcover, tends to be replaced with the word "quasicompact", because this property is possessed by most of the spaces under study, even if they are not what we are used to thinking of as compact. Fore example, $\operatorname{Spec}\mathbb{C}[x]$, the algebraic-geometric model of the topological space $\mathbb{C}$, is quasicompact, even though it is not compact in the Euclidean topology. There are other more advanced concepts that do a better job of substituting for the usual notion of compactness.)

(g) More generally, each X_f is quasicompact.

(h) An open subset of X is quasicompact if and only if it is a finite union of X_f's.

4. Exercise 18. Let $x \in \operatorname{Spec} A$ be a point of $\operatorname{Spec} A$ the topological space, and let $\mathfrak{p}_x$ be the same element of $\operatorname{Spec} A$ except stressing that it is a prime ideal of A.

   (a) Show $\{x\} \subset \operatorname{Spec} A$ is closed if and only if $\mathfrak{p}_x$ is maximal.

   (b) Show the closure of $\{x\}$ is $V(\mathfrak{p}_x)$.

   (c) $y \in \overline{\{x\}} \Leftrightarrow \mathfrak{p}_y \supset \mathfrak{p}_x$.

   (d) X is a T_0 space, i.e. any two points are separated by an open set containing one and not the other.

5. Exercise 21. Let $\phi : A \to B$ be a ring homomorphism. Let $X = \operatorname{Spec} A, Y = \operatorname{Spec} B$. If $\mathfrak{q} \in Y$, then $\phi^{-1}(\mathfrak{q})$ is a prime ideal of A, i.e. a point of X. So ϕ induces a mapping $\phi^* : Y \to X$. (This map is called the *pullback* of ϕ.) Show that

   (a) If $f \in A$ then $\phi^{*-1}(X_f) = Y_{\phi(f)}$, and thus that ϕ^* is continuous.

   (b) If $\mathfrak{a}$ is an ideal of A, then $\phi^{*-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$.

   (c) If $\mathfrak{b} \lhd B$, then $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^c)$.

   (d) If ϕ is surjective, then ϕ^* is a homeomorphism of Y onto the closed subset $V(\ker\phi)$ of X. (In particular, $\operatorname{Spec} A$ and $\operatorname{Spec} A/\mathfrak{N}$ are naturally homeomorphic.)

(e) If ϕ is injective, then $\phi^*(Y)$ is dense in X. More generally, $\phi^*(Y)$ is dense in $X \Leftrightarrow \ker\phi \subset \mathfrak{N}$.

(f) Let $\psi : B \to C$ be another ring homomorphism. Then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

(g) Let A be an integral domain with just one non-zero prime ideal $\mathfrak{p}$, and let K be A's field of fractions. Let $B = A/\mathfrak{p} \times K$. Define $\phi : A \to B$ by $\phi(x) = (\bar{x}, x)$, where $\bar{x}$ is the image of x in $A/\mathfrak{p}$. Show that ϕ^* is bijective but not a homeomorphism.

1.3 Affine Varieties

1. Exercise 26. Here Atiyah and MacDonald define MaxSpec (the set of maximal ideals), noting that in general it does not have the nice functorial properties of Spec, because maximal ideals don't always pull back to maximal ideals. But in some cases it is useful because the elements of MaxSpec can be identified with the points of a topological space.

   Let X be a compact hausdorff topological space and let $C(X)$ be the ring of continuous real-valued functions on X. For $x \in X$, let $\mathfrak{m}_x$ be the ideal of functions vanishing at x. It is maximal because it is the kernel of the homomorphism $C(X) \to \mathbb{R}$ that maps $f \mapsto f(x)$, and this homomorphism is surjective with image the field $\mathbb{R}$. So $x \mapsto \mathfrak{m}_x$ is a mapping μ of X into $\tilde{X} = \operatorname{MaxSpec} C(X)$. The problem aims to show μ is a homeomorphism.

   (a) Show that μ is surjective: in other words, every maximal ideal of $C(X)$ has the form $\mathfrak{m}_x$.

   (b) By Urysohn's lemma, the continuous functions separate the points of x. Thus show μ is injective.

   (c) Let $f \in C(X)$. Let $U_f = \{x \in X : f(x) \neq 0\}$. (I feel Atiyah and MacDonald could have called this X_f to stress the connection with the notation in Exercises 17 and 21.) Let $\tilde{U}_f = \{\mathfrak{m} \in \tilde{X} : f \notin \mathfrak{m}\}$. Show that $\mu(U_f) = \tilde{U}_f$. Show that the open sets U_f, resp. $\tilde{U}_f$, form a basis for the topology of X, resp. $\tilde{X}$, and thus μ is a homeomorphism. (This is a motivating example for algebraic geometry because it shows that the geometric structure of X can be recovered from the ring $C(X)$.)

   Thus X can be reconstructed as a topological space from $C(X)$.

2. Exercise 27. Let k be an algebraically closed field and let

$$f_\alpha(t_1, \dots, t_n) = 0$$

be a set of polynomial equations (indexed by α) in n variables, with coefficients in k. The set X of all points $x = (x_1, \ldots, x_n) \in k^n$ which satisfy these equations is an *affine algebraic variety.*

Consider the set of all polynomials $g \in k[t_1, \ldots, t_n]$ with the property that $g(x) = 0$ for all $x \in X$. Check that this set is an ideal $I(X)$ in the polynomial ring. It is called the *ideal of the variety* X. The quotient ring

$$k[X] = k[t_1, \ldots, t_n]/I(X)$$

is the ring of polynomial functions on X, because two polynomials g, h define the same function on X if and only if $g - h$ vanishes at every point of X, that is, if and only if $g - h \in I(X)$.

Let ξ_i be the image of t_i in $k[X]$. The ξ_i (for $1 \le i \le n$) are the *coordinate functions* on X: if $x \in X$, then $\xi_i(x)$ is the ith coordinate of x. $k[X]$ is generated as a k-algebra by the coordinate functions, so is called the *coordinate ring* (or affine algebra) of X.

As in Exercise 26, for each $x \in X$ let $\mathfrak{m}_x$ be the ideal of all $f \in k[X]$ such that $f(x) = 0$; check that it is a maximal ideal of $k[X]$. Hence, if $\tilde{X} = \mathrm{MaxSpec}(k[X])$, we have defined a mapping $\mu : X \to \tilde{X}$, namely $x \mapsto \mathfrak{m}_x$.

It is easy to show that μ is injective: if $x \ne y$, we must have $x_i \ne y_i$ for some i $(1 \le i \le n)$, and hence $\xi_i - x_i$ is in $\mathfrak{m}_x$ but not in $\mathfrak{m}_y$, so that $\mathfrak{m}_x \ne \mathfrak{m}_y$. What is less obvious (but still true) is that μ is *surjective.* This is one form of Hilbert's Nullstellensatz (see chapter 7).

3. Exercise 28. Let $f_1, \ldots, f_m$ be elements of $k[t_1, \ldots, t_n]$. They determine a *polynomial mapping* $\phi : k^n \to k^m$: if $x \in k^n$, the coordinates of $\phi(x)$ are $f_1(x), \ldots, f_m(x)$.

   Let X, Y be affine algebraic varieties in k^n, k^m respectively. A mapping $\phi : X \to Y$ is said to be *regular* if ϕ is the restriction to X of a polynomial mapping from k^n to k^m.

   If η is a polynomial function on Y, then $\eta \circ \phi$ is a polynomial function on X. Hence ϕ induces a k-algebra homomorphism $k[Y] \to k[X]$, namely $\eta \mapsto \eta \circ \phi$. Show that in this way we obtain a one-to-one correspondence between regular mappings $X \to Y$ and k-algebra homomorphisms $k[Y] \to k[X]$.